CDI - II - Prática - 26/5/21

Ficha 11 + Ficha 12

Ficha 11 - Potenciais escalares

→ Campos conservativos.

→ Teorema Fundamental do Cálculo

$F = \nabla \varphi$ F é conservativo

$\varphi : \mathbb{R}^n \to \mathbb{R}$ potencial escalar de F.

- Se $F = \nabla\varphi$ então F é fechado

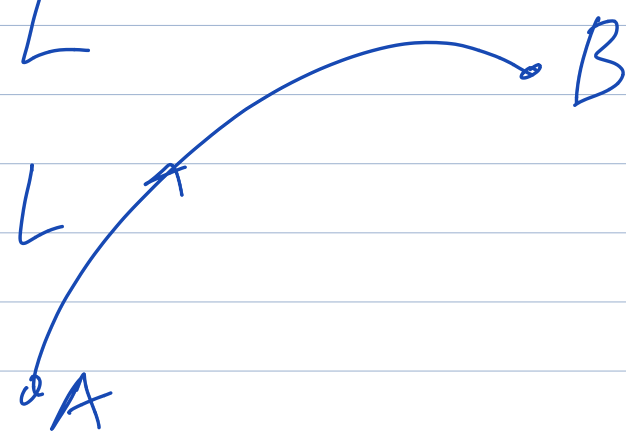
$$\frac{\partial F_k}{\partial x_j} = \frac{\partial F_j}{\partial x_k}, \quad j \neq k$$

- $R(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

R é fechado mas não é conservativo.

- T.F.C. : se $F = \nabla\varphi$ então

$$\int_L F = \varphi(B) - \varphi(A)$$



- $F = \nabla\varphi$ sse $\oint_L F = 0 \quad \forall L \subset \mathbb{R}^n$ fechada.

3 - a) $2y = \frac{\partial a_1}{\partial y} \neq \frac{\partial a_2}{\partial x} = 3x^2$

a não é fechado $\Rightarrow$ não é conservativo.

3 - b) $b(x,y) = (x^3, y^2) + (y, x)$

$$\varphi(x,y) = \frac{x^4}{4} + \frac{y^3}{3} + xy + C$$

$C \in \mathbb{R}$

$\nabla \varphi = b$ ✓

3 - c) $\varphi(x,y) = e^x + e^y + C$, $C \in \mathbb{R}$. ✓

3 - d) $d(x,y) = \frac{1}{2}\left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$

$$\varphi(x,y) = \frac{1}{2}\log(x^2+y^2) + C$$ ✓

3-e) $e(x,y,z) = (y, x, 2z)$

$= (y, x, 0) + (0, 0, 2z)$

$\varphi(x,y,z) = xy + z^2 + C$

$\boxed{\nabla\varphi = e}$ ✓

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = y \longrightarrow \varphi(x,y,z) = xy + A(y,z) \\ \dfrac{\partial\varphi}{\partial y} = x \longrightarrow \cancel{x} + \dfrac{\partial A}{\partial y} = \cancel{x} \rightarrow \dfrac{\partial A}{\partial y} = 0 \\ \dfrac{\partial\varphi}{\partial z} = 2z \longrightarrow B'(z) = 2z \end{cases}$$

$B(z)$

$A(y,z) = B(z)$

$B(z) = z^2 + C$

$\rightarrow \varphi(x,y,z) = xy + z^2 + C$

$$(y, x) = \nabla(xy)$$ ✓

$$(f(x), g(y), h(z)) = \nabla\left(P_{f(x)} + P_{g(y)} + P_{h(z)}\right)$$ ✓

$$\nabla\varphi(x,y) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = (y, x)$$

$$\varphi(x,y) = xy$$

$$F(x,y) = (x^4, y^3)$$

$$F = \nabla\varphi \qquad \varphi(x,y) = \frac{x^5}{5} + \frac{y^4}{4} + C$$

3-f) $\dfrac{\partial g_1}{\partial y} = -1 \neq \dfrac{\partial g_2}{\partial x} = 1$

g não é fechado $\Rightarrow$ g não é conservativo

g não tem potencial escalar.

— // —

3-g) $f(x,y,z) = (1, 2, 3)$

$\varphi(x,y,z) = x + 2y + 3z + c$

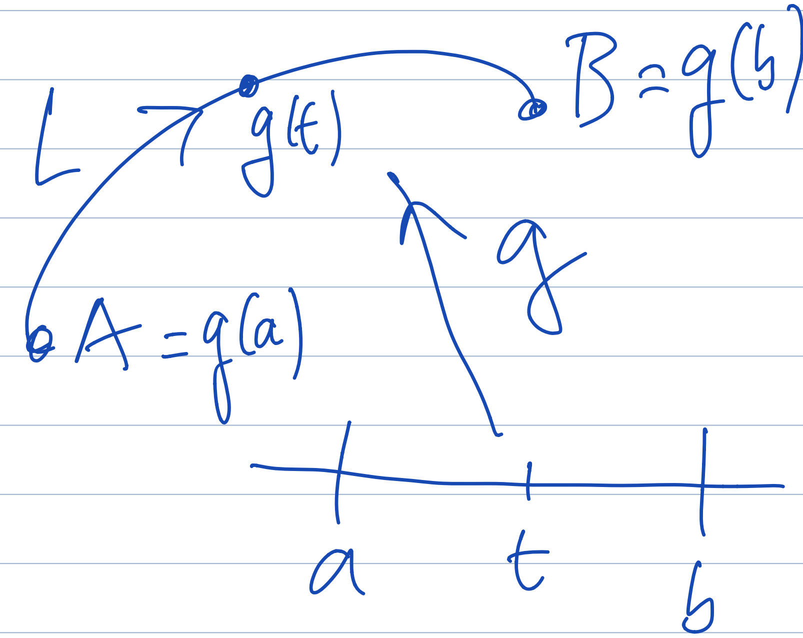
— // —

4 – $F(x,y,z) = \frac{1}{2}\left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 0\right) +$

$$+ (0, 0, 2z)$$

$$\varphi(x,y,z) = \frac{1}{2}\log(1+x^2+y^2) + z^2 + C$$

$$\int_L F \cdot dg = \varphi(B) - \varphi(A)$$



4-a) $A = g(0) = (1,0,0)$

$B = g(2\pi) = (1,0,2\pi)$

$$\varphi(B) = \frac{1}{2}\log 2 + 4\pi^2 + C$$

$$\varphi(A) = \frac{1}{2}\log 2 + C$$

$$\varphi(B) - \varphi(A) = 4\pi^2 \quad //.$$

4-b) $\begin{cases} y^2 + z^2 = 1 \\ x = y^2 - z^2 \end{cases} \longrightarrow \begin{array}{l} y = \cos t \\ z = \operatorname{sen} t \\ x = \cos^2 t - \operatorname{sen}^2 t \end{array}$

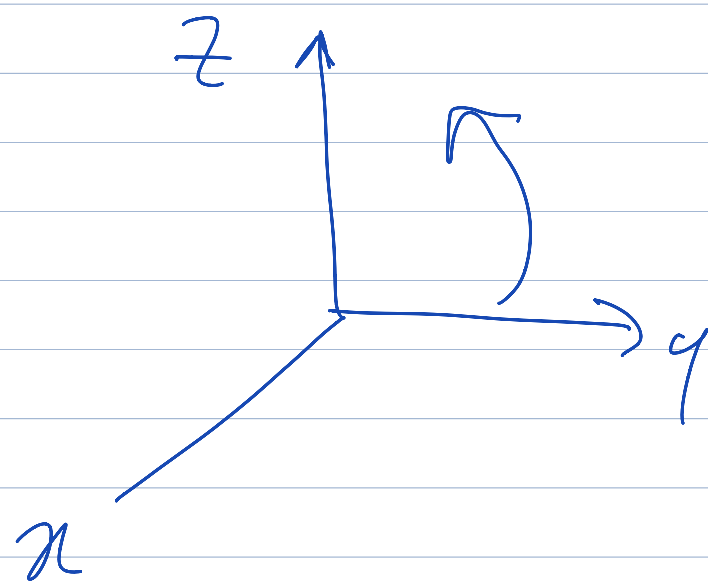
$$g(t) = (\cos^2 t - \operatorname{sen}^2 t, \cos t, \operatorname{sen} t)$$

$$0 \leq t \leq 2\pi.$$

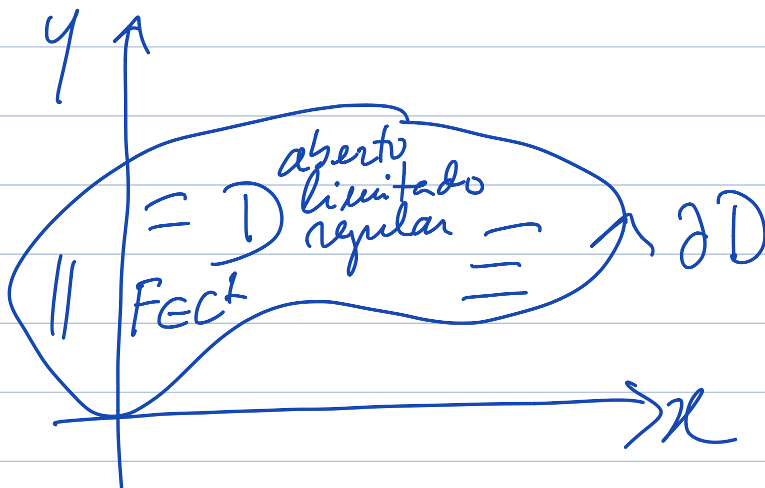
$$A = g(0) = (1,1,0)$$

$$B = g(2\pi) = (1,1,0)$$

$$\varphi(B) - \varphi(A) = 0 \ !$$



Ficha 12: T. de Green ($\mathbb{R}^2$)

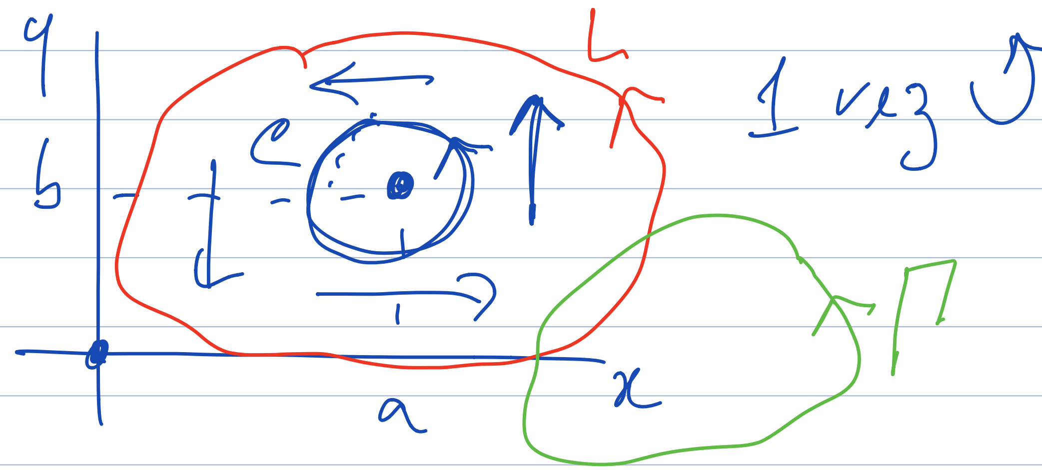


$F = (P, Q)$

$F: \mathbb{R}^2 \to \mathbb{R}^2$, C^1 em D

$$\underbrace{\int_{\partial D} F \cdot dg}_{\text{Trabalho de } F \text{ na fronteira de } D.} = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} dx\,dy$$

"Ralo geral"



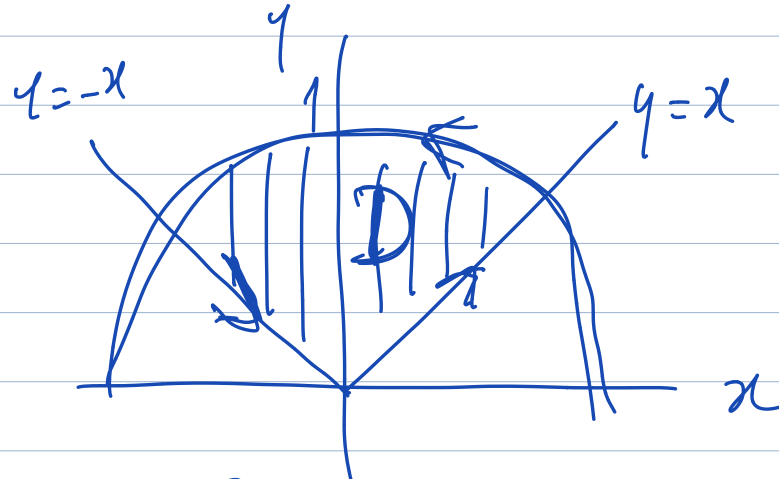
$$R(x,y) = \left(-\frac{y-b}{(x-a)^2+(y-b)^2}, \frac{x-a}{(x-a)^2+(y-b)^2} \right)$$

$$\oint_C R \cdot dq = 2\pi$$

$$\oint_L R \cdot dq = 2\pi \quad (\text{Green})$$

$$\oint_\Gamma R \cdot dq = 0 \quad !$$

1 —



$$\int_{\partial D} F = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=3} dx\,dy$$

$$= 3\,\mathrm{vol}_2(D)$$

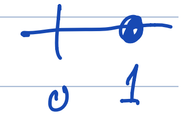
$$= 3\,\frac{1}{4}\pi = \frac{3\pi}{4}$$

$$F(x,y) = \big(P(x,y),\, Q(x,y)\big)$$

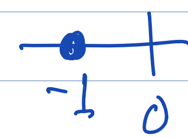
$$= (-2y,\, x)$$

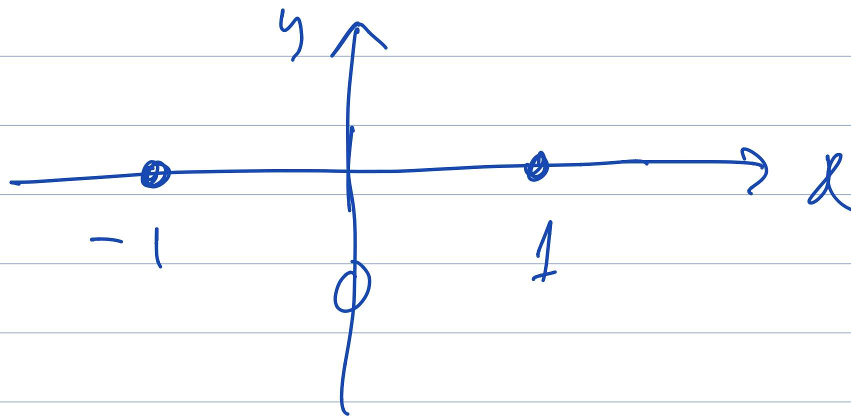
2- $F = G + H$

$$G(x,y) = \left(-\frac{y}{(x-1)^2+y^2} , \frac{x-1}{(x-1)^2+y^2} \right)$$

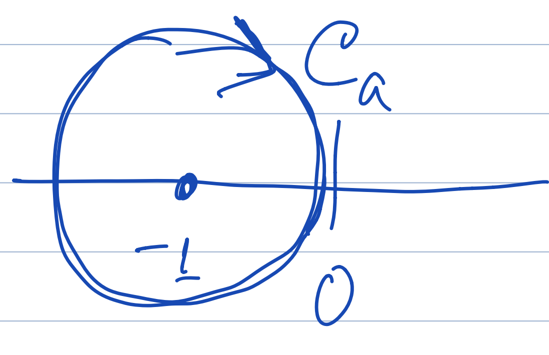


$$H(x,y) = \left(\frac{y}{(x+1)^2+y^2} , \frac{-(x+1)}{(x+1)^2+y^2} \right)$$





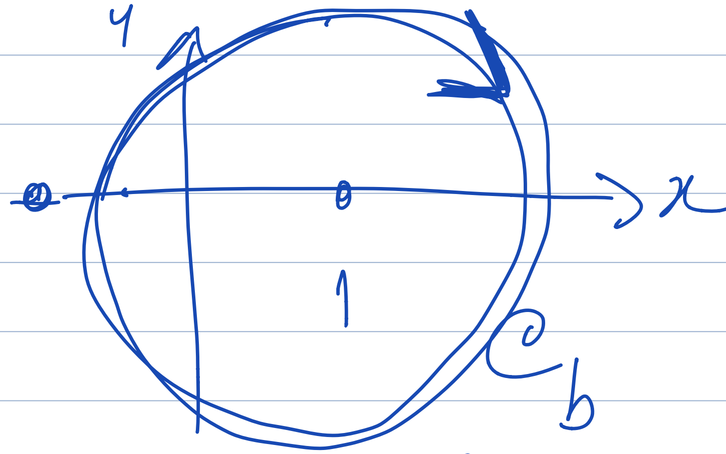
a)



$$\oint_{C_a} G = 0$$

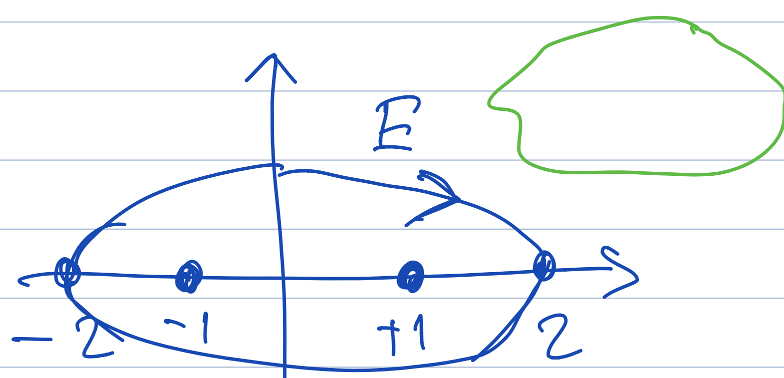
$$\oint_{C_a} H = 2\pi$$

2-b)



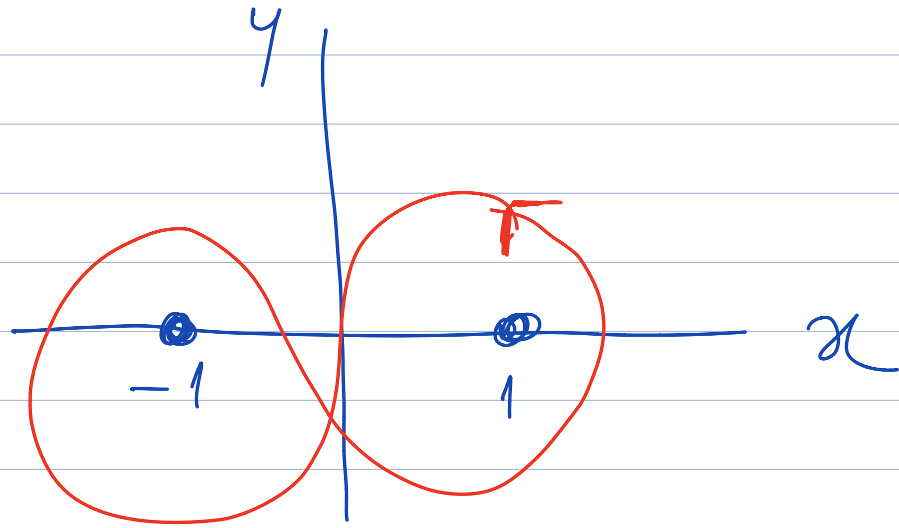
$$\oint_{C_b} G = -2\pi \quad ; \quad \int_{C_b} H = 0$$

2-c)



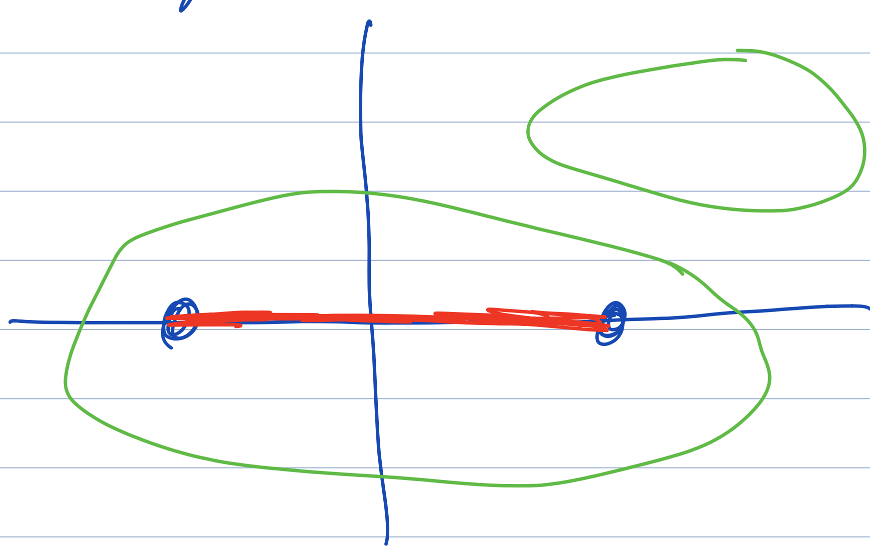
$$\oint_E G = -2\pi \quad ; \quad \oint_E H = 2\pi$$

$$\oint_E F = 0 .$$



$$\mathbb{R}^2 \setminus \{(x,y) : y=0 ; -1 \le x \le 1\}$$

F é gradiente neste conjunto



3 - $$\int_{\partial D} F = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{c \in \mathbb{R}} dx\,dy$$

$$= c\,\mathrm{Vol}_2(D), \quad c \neq 0$$

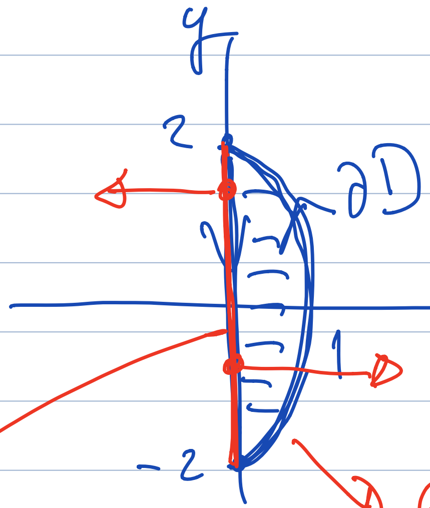
$$\boxed{\mathrm{Vol}_2(D) = \frac{1}{c}\int_{\partial D} F.}$$

1 - $F(x,y) = (-y, x) \longrightarrow c = 2$

$\boxed{2 - F(x,y) = (-y, 0) \longrightarrow c = 1}$

3 - $F(x,y) = (0, x) \longrightarrow c = 1$

$F(x,y) = (-y, 0)$



$g(t) = (\cos t, 2\sin t)$

$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

F normal ao segmento de recta

$W = 0.$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(g(t)) \cdot g'(t)\, dt$$

etc..

$= \pi$